COMPUTING QUOTIENTS OF THE BRUHAT-TITS TREE FOR $\mathrm{GL}_2(\mathbf{Q}_p)$ BY QUATERNIONIC GROUPS

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ABSTRACT. In this paper we describe an algorithm for computing certain quaternionic quotients of the Bruhat-Tits tree for $\mathrm{GL}_2(\mathbf{Q}_p)$. These quotients are of arithmetic interest as they describe bad fibers of integral models of Shimura curves.

CONTENTS

1. Introduction	1
2. The Bruhat-Tits tree	2
2.1. Definition	2
2.2. Quaternion algebras and subgroups of interest	4
3. Computing quotients	5
3.1. Solving the Γ-equivalence problem	5
3.2. Computing quotients of the tree	7
4. Examples	10
References	12

1. INTRODUCTION

The goal of this paper is to describe an algorithm for computing quotients of the Bruhat-Tits tree for $\mathrm{GL}_2(\mathbf{Q}_p)$ by discrete subgroups of $\mathrm{GL}_2(\mathbf{Q}_p)$ arising from definite quaternion algebras over $\mathbf{Q}$ which are unramified at p. Such quotients are of arithmetic interest as, by work of Cerednik and Drinfeld (see [BC91] for a detailed account), they describe special fibers of integral models of Shimura curves at primes of bad reduction.

In [Gre06, Appendix], Matthew Greenberg describes an algorithm for computing quotients of the Bruhat-Tits tree arising from subgroups of a maximal order in the rational Hamilton quaternions. This allows him to describe Shimura curves attached to *indefinite* quaternion algebras over $\mathbf{Q}$ of conductor $2p$, for p an odd prime. Greenberg's method relies crucially on the fact that the quotients under consideration have exactly two vertices. As this is not always the case for other quaternion algebras, it is desirable to have a more general method.

Our algorithm treats arithmetic groups arising from arbitrary Eichler orders within arbitrary quaternion algebras over $\mathbf{Q}$. More precisely, the results in this paper prove the following result.

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Theorem 1.1. *Let p be a finite prime and let $\mathcal{T}$ denote the Bruhat-Tits tree for $\mathrm{GL}_2(\mathbf{Q}_p)$. Let $B/\mathbf{Q}$ denote a definite quaternion algebra of discriminant N^- which is split at the finite prime p. Let N^+ be a positive integer coprime to pN^-. Let $R \subseteq B$ be an Eichler $\mathbf{Z}$-order of level N^+, and let Γ denote the subgroup of elements of reduced norm 1 in $R[1/p]$. Let Γ act on $\mathcal{T}$ via a splitting*

$$\iota \colon B \otimes_{\mathbf{Q}} \mathbf{Q}_p \xrightarrow{\sim} M_2(\mathbf{Q}_p).$$

Then there exists an explicit algorithm for computing the quotient $\Gamma\backslash\mathcal{T}$ in time

$$O\left(\frac{(\log g)^3 g(p^3+2g)}{(\log p)^3 p^2}\right),$$

where g is the genus of the Shimura curve attached to R or, equivalently, g is the first Betti number of the graph $\Gamma\backslash\mathcal{T}$.

In particular, for fixed p our algorithm runs in time

$$O\left(g^2(\log g)^3\right).$$

We expect that the algorithm of Theorem 1.1, described in Subsection 3.2, will generalize to quaternion algebras over totally real fields.

Our interest in developing this algorithm stemmed from a desire to use the quotient, and the work of Darmon-Pollack in [PS], to compute values of rigid analytic modular forms and anti-cyclotomic p-adic L-functions. We will explain these and other applications in a forthcoming paper.

This note is arranged as follows: in Section 2 we introduce notation and describe precisely which quotients of the Bruhat-Tits tree are treated in this paper. In Section 3 we describe the algorithm and give an estimate for its complexity. In the final Section 4 we show some examples of quotients. These examples were computed using a *Sage* (see [S$^+$11]) implementation of the algorithm, and the code is available on the second author's website.

Note that Böckle and Butenuth [BB10] have developed an algorithm for computing quaternionic quotients of the Bruhat-Tits tree for $\mathrm{GL}_2(\mathbf{F}_q[\![T]\!])$. The authors wish to thank them for several helpful discussions about their work. The authors also wish to thank Henri Darmon, Matthew Greenberg, Victor Rotger and John Voight for their advice and encouragement.

2. The Bruhat-Tits tree

In this paper p denotes a fixed rational prime.

2.1. **Definition.** The Bruhat-Tits tree $\mathcal{T}$ for $\mathrm{GL}_2(\mathbf{Q}_p)$ is a graph described as follows: the vertices of $\mathcal{T}$ are the homothety classes of $\mathbf{Z}_p$-lattices in $\mathbf{Q}_p^2$, where we regard $\mathbf{Q}_p^2$ as a space of column vectors. We write $V(\mathcal{T})$ for the set of vertices of $\mathcal{T}$. Two vertices are joined by an unordered edge if there exist representative lattices Λ_1 and Λ_2 for the respective vertices such that

$$p\Lambda_1 \subsetneq \Lambda_2 \subsetneq \Lambda_1.$$

If Λ is a lattice in $\mathbf{Q}_p^2$ then we write $[\Lambda]$ for the corresponding homothety class. We let $E(\mathcal{T})$ denote the set of ordered pairs of adjacent vertices of $\mathcal{T}$ and we refer to elements of $E(\mathcal{T})$ as ordered edges of $\mathcal{T}$.

Proposition 2.1. *The Bruhat-Tits tree for $\mathrm{GL}_2(\mathbf{Q}_p)$ is a connected tree such that each vertex has degree $p+1$.*

Proof. See [DT08, Section 1.3.1, Proposition 8]. □

The group $\mathrm{GL}_2(\mathbf{Q}_p)$ acts on $\mathcal{T}$: the action on vertices is given by multiplication of column vectors by matrices in $\mathrm{GL}_2(\mathbf{Q}_p)$. This preserves the adjacency relation described above, and so it thus describes an action on $\mathcal{T}$ by graph automorphisms. If $A \in \mathrm{GL}_2(\mathbf{Q}_p)$ then we let $[A]$ denote homothety class of the lattice in $\mathbf{Q}_p^2$ which is spanned by the columns of A. The map $A \mapsto [A]$ induces a bijection

$$\mathrm{GL}_2(\mathbf{Q}_p)/\mathbf{Q}_p^\times \mathrm{GL}_2(\mathbf{Z}_p) \xrightarrow{\sim} V(\mathcal{T})$$

which is $\mathrm{GL}_2(\mathbf{Q}_p)$-equivariant for the natural left actions given by matrix multiplication.

We would like a similar description for the oriented edges of $\mathcal{T}$. For this we consider the oriented edge (v_0, v_1) of $\mathcal{T}$ from the vertex v_0 corresponding to the homothety class of $\mathbf{Z}_p^2$, to the vertex

$$v_1 = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} v_0.$$

The stabilizer of v_1 for the action of $\mathrm{GL}_2(\mathbf{Q}_p)$ is precisely

$$\mathbf{Q}_p^\times \cdot \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \mathrm{GL}_2(\mathbf{Z}_p) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}^{-1}.$$

Thus, if we write

$$\begin{aligned} \Gamma_0(p) &= \mathrm{GL}_2(\mathbf{Z}_p) \cap \left(\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \mathrm{GL}_2(\mathbf{Z}_p) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}^{-1} \right) \\ &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z}_p) \;\middle|\; p \mid c \right\}, \end{aligned}$$

then the stabilizer of (v_0, v_1) for the action of $\mathrm{GL}_2(\mathbf{Q}_p)$ is $\mathbf{Q}_p^\times \cdot \Gamma_0(p)$. The map

$$A \mapsto \left([A], \left[A \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \right] \right)$$

yields a bijection

$$\mathrm{GL}_2(\mathbf{Q}_p)/\mathbf{Q}_p^\times \cdot \Gamma_0(p) \xrightarrow{\sim} E(\mathcal{T}) \tag{2.1}$$

which is equivariant for the left action of $\mathrm{GL}_2(\mathbf{Q}_p)$. We will refer to the vertx v_0 as the *privileged vertex* and to the edge (v_0, v_1) as the *privileged edge*.

Lemma 2.2. *There is a set of coset representatives $\{e_i\}_i$ for $\mathrm{GL}_2(\mathbf{Q}_p)/\mathbf{Q}_p^\times \cdot \Gamma_0(p)$ given by matrices with coefficients in $\mathbf{Z}$. Moreover, there is an effective algorithm that, given any matrix in $g \in \mathrm{GL}_2(\mathbf{Q}_p)$, finds a corresponding scalar $\lambda \in \mathbf{Q}_p^\times$ and matrix $t \in \Gamma_0(p)$ satisfying:*

$$g\lambda t = e_i.$$

An analogous statement holds for $\mathrm{GL}_2(\mathbf{Q}_p)/\mathbf{Q}_p^\times \cdot \mathrm{GL}_2(\mathbf{Z}_p)$.

Proof. The matrices e_i have the form:

$$\begin{pmatrix} p^a & 0 \\ r & p^b \end{pmatrix}, \quad 0 \le r \le p^{b+1},$$

$$\begin{pmatrix} 0 & p^a \\ p^b & r \end{pmatrix}, \quad 0 \le r \le p^b.$$

It is easy to see what elementary matrix transformations need to be performed in order to reduce any matrix in $M_2(\mathbf{Q}_p)$ to one in the above form. □

One can encode the vertices and edges of the Bruhat-Tits tree conveniently via a set of representatives as in Lemma 2.2.

Remark 2.3. It is crucial to have an efficient reduction algorithm, as in Lemma 2.2, if one wants to encode the Γ-action on $\mathcal{T}$. The action of Γ on $\mathcal{T}$ is induced from the action on $M_2(\mathbf{Q}_p)$ by matrix multiplication, and in order to descend this to a computable action on the tree, one needs to calculate the reduction of γx for any $\gamma \in \Gamma$ and $x \in M_2(\mathbf{Q}_p)$. This will be used in Algorithm 1. Note also that the running time of such a reduction can be regarded as bounded by a constant, and therefore it can be ignored in the complexity estimates.

2.2. **Quaternion algebras and subgroups of interest.** Let $B/\mathbf{Q}$ denote a definite quaternion algebra which is unramified at p. If l is a place of $\mathbf{Q}$, then we write $B_l = B \otimes_{\mathbf{Q}} \mathbf{Q}_l$ where $\mathbf{Q}_l$ is the completion of $\mathbf{Q}$ at l. Thus B_∞ is isomorphic with the Hamilton quaternions and $B_p \cong M_2(\mathbf{Q}_p)$. Let N^- denote the discriminant of B, so that N^- is a product of an odd number of distinct primes, and N^- is coprime to p.

Let $R^{\max} \subseteq B$ denote a maximal $\mathbf{Z}[1/p]$-order. Let N^+ be a positive integer which is coprime to pN^-, and let $R \subseteq R^{\max}$ denote an Eichler $\mathbf{Z}[1/p]$-order of level N^+. Since B is unramified at p, it satisfies the Eichler condition for $\mathbf{Z}[1/p]$-orders and there hence exists a unique, up to conjugation by $B^\times$, Eichler $\mathbf{Z}[1/p]$-order of each level. Let ι denote an isomorphism

$$\iota \colon B_p \cong M_2(\mathbf{Q}_p)$$

which satisfies $\iota(R_p^{\max}) = M_2(\mathbf{Z}_p)$. Let $\Gamma = \Gamma(p, N^-, N^+)$ denote the subgroup of elements of reduced norm 1 in R. The group Γ acts on the Bruhat-Tits tree via the splitting ι.

A discrete subgroup of $\mathrm{GL}_2(\mathbf{Q}_p)$ is said to be *Schottky* if it acts without fixed points on the vertices $V(\mathcal{T})$ of the Bruhat-Tits tree.

Proposition 2.4. *(1) The group Γ is a finitely generated discrete subgroup of $\mathrm{GL}_2(\mathbf{Q}_p)$ and the quotient $\Gamma\backslash\mathcal{T}$ is finite.*

(2) There exists an integer constant $M \geq 1$ depending only on pN^- such that if $N^+ \geq M$, the group Γ is Schottky. In this case the abelianization of Γ is a finite free $\mathbf{Z}$-module of rank

$$g = 1 - V + E,$$

where V and E denote respectively the number of vertices and edges of $\Gamma\backslash\mathcal{T}$.

Proof. These are standard facts from the theory of Schottky groups. See, for example, [GvdP80, Section I.3]. □

Remark 2.5. Our interest in such subgroups arises from the fact that quotients of the form $\Gamma\backslash\mathcal{T}$ describe bad special fibers of Shimura curves, cf. [BC91]. In the Schottky case the corresponding special fiber has irreducible components in bijection with the vertices of $\Gamma\backslash\mathcal{T}$. Each component is isomorphic with $\mathbf{P}^1$ over $\mathbf{F}_p$, and two components meet in an ordinary double points defined over $\mathbf{F}_p$ if and only if the corresponding vertices of $\Gamma\backslash\mathcal{T}$ are joined by an edge. Thus, the algorithm in this paper describes a way to compute the bad special fibers of integral models of Shimura curves over $\mathbf{Q}$.

3. Computing quotients

3.1. **Solving the Γ-equivalence problem.** Let $B/\mathbf{Q}$ denote a definite quaternion algebra which is unramified at p, as in Section 2.2. Let N^- be the discriminant of B, so that N^- is the product of an odd number of distinct prime numbers. Let N^+ be an integer which is relatively prime to pN^-. Let R, ι and Γ be as defined in Section 2.2.

Although Γ is a finitely generated group, it is not defined in terms of a convenient finite presentation. There is, however, a natural increasing filtration of Γ by finite sets which can be used to compute a finite generating set. For each $n \geq 0$ set

$$\Gamma_n = \left\{ \iota\left(\frac{x}{p^n}\right) \;\middle|\; x \in R \text{ and } \operatorname{nrd}(x) = p^{2n} \right\}.$$

Then $\Gamma_n \subseteq \Gamma_{n+1}$ since $x/p^n = (px)/p^{n+1}$ and

$$\Gamma = \bigcup_{n\geq 0} \Gamma_n.$$

Note that each Γ_n is indeed a finite set since B is definite.

Following [BB10], we introduce some generic notation for group actions. If G is a group and X a left G-set, and if u and v are elements of X, then we write $\operatorname{Hom}_G(u,v)$ to denote the collection of all elements of G which map u into v. We also write $\operatorname{Stab}_G(u) = \operatorname{Hom}_G(u,u)$.

The central problem that arises when computing the quotient $\Gamma\backslash\mathcal{T}$ is the computation of the sets $\operatorname{Hom}_\Gamma(u,v)$ for two vertices u and v of $\mathcal{T}$ (or for two ordered edges). Our solution to this problem uses short vector searches in certain $\mathbf{Z}$-lattices of rank 4. In the following exposition of our method we concentrate on the case of vertices. The case of ordered edges is treated analogously.

Assume that two vertices u and v of $\mathcal{T}$ are represented by reduced matrices as in Lemma 2.2. We will abuse notation and denote the representing matrices also as u and v. Define m via the formula $2m = \operatorname{val}_p(\det u \det v)$. The integer m is the half-length of the path that joins the vertex u to the vertex v passing through the privileged vertex v_0. Write $p^a = \det u$ and $p^b = \det v$, so that $2m = a + b$. Note that

$$\begin{aligned} \operatorname{Hom}_\Gamma(u,v) &= \operatorname{Hom}_{\mathrm{GL}_2(\mathbf{Q}_p)}(u,v) \cap \Gamma \\ &= \left(v \operatorname{Stab}_{\mathrm{GL}_2(\mathbf{Q}_p)}(v_0) u^{-1}\right) \cap \Gamma \\ &= \left(v \mathbf{Q}_p^\times \mathrm{GL}_2(\mathbf{Z}_p) u^{-1}\right) \cap \Gamma. \end{aligned} \tag{3.1}$$

Lemma 3.1. *If m is not an integer then $\operatorname{Hom}_\Gamma(u,v) = \emptyset$. Otherwise one has*

$$\operatorname{Hom}_\Gamma(u,v) = p^{-m} v M_2(\mathbf{Z}_p) u^* \cap \Gamma,$$

where u^* is the matrix satisfying $uu^* = \det u$.

Proof. Since $\Gamma \subseteq \mathrm{SL}_2(\mathbf{Q}_p)$, the corollary to Proposition 1 of [Ser03, Chapter 2, Subsection 1.2] shows that m must be an integer for u and v to be equivalent under Γ, as the group Γ preserves the parity of the distance between any two vertices.

By Equation (3.1), it suffices to show that

$$\left(v \mathbf{Q}_p^\times \mathrm{GL}_2(\mathbf{Z}_p) u^{-1}\right) \cap \Gamma = p^{-m} v M_2(\mathbf{Z}_p) u^* \cap \Gamma.$$

Write $z \in (v\mathbf{Q}_p^\times \operatorname{GL}_2(\mathbf{Z}_p)u^{-1}) \cap \Gamma$ as $z = v\lambda g u^{-1}$ for $\lambda \in \mathbf{Q}_p^\times$ and $g \in \operatorname{GL}_2(\mathbf{Z}_p)$. Observe that $z = p^{-a}v\lambda g u^*$. Since $z \in \Gamma$ we have $\det z = 1$ and hence

$$\lambda^2 p^{b-a}\sigma = 1,$$

where $\sigma = \det g \in \mathbf{Z}_p^\times$. Therefore $2\operatorname{val}_p(\lambda) = a - b$ and thus $\operatorname{val}_p(p^{-a}\lambda) = -m$. We conclude that $p^{-a}\lambda g$ belongs to $p^{-m}\operatorname{GL}_2(\mathbf{Z}_p) \subseteq p^{-m}M_2(\mathbf{Z}_p)$, and thus

$$\left(v\mathbf{Q}_p^\times \operatorname{GL}_2(\mathbf{Z}_p)u^{-1}\right) \cap \Gamma \subseteq p^{-m}vM_2(\mathbf{Z}_p)u^* \cap \Gamma.$$

Conversely if one writes $z \in p^{-m}vM_2(\mathbf{Z}_p)u^* \cap \Gamma$ as

$$z = p^{-m}vgu^* = p^{a-m}vgu^{-1},$$

for $g \in M_2(\mathbf{Z}_p)$, then one checks that $\det g = 1$ and hence $g \in \operatorname{GL}_2(\mathbf{Z}_p)$. The lemma follows. □

Remark 3.2. Since p fixes every homothety class of lattices, it acts trivially on the vertices of $\mathcal{T}$. Thus, the equation in the preceding lemma can be rescaled to read:

$$\operatorname{Hom}_\Gamma(u,v) = \bigcup_{n \geq 0} p^{n-m}vM_2(\mathbf{Z}_p)u^* \cap p^n\Gamma_n.$$

This expression realizes $\operatorname{Hom}_\Gamma(u,v)$ inside the Eichler order R. This is convenient, as although Γ consists of quaternions of unit norm, in practice one usually works with integral elements in R of norm an even power of p.

Since Γ is defined via the local splitting ι, explicit knowledge of ι is required to compute effectively with Γ. Of course, ι is a linear map between four-dimensional vector spaces, and so it can be described easily as a matrix. A complicating factor arises from the fact that this matrix can only be stored up to a finite precision. In particular, the image of ι is not known explicitly, which complicates the problem of determining if a given matrix in $M_2(\mathbf{Q}_p)$ belongs to Γ. In order to bypass this difficulty we introduce the notion of an approximation to a $\mathbf{Q}_p$-linear map.

Definition 3.3. Let $n \geq 0$ be an integer, let V and W be two finite dimensional $\mathbf{Q}_p$-vector spaces, and let $\Lambda_V \subseteq V$ and $\Lambda_W \subseteq W$ be $\mathbf{Z}_p$-lattices. Let $f\colon V \to W$ be a $\mathbf{Q}_p$-linear map satisfying $f(\Lambda_V) \subseteq \Lambda_W$. Then an *approximation* of f to precision n is a $\mathbf{Q}_p$-linear map $g\colon V \to W$ such that $g \equiv f \pmod{p^n}$ when restricted to Λ_V.

The following lemma lies at the heart of our algorithm.

Lemma 3.4. *Let u and v be matrices in $M_2(\mathbf{Z}) \cap \operatorname{GL}_2(\mathbf{Q}_p)$ representing two vertices of $\mathcal{T}$. Let $f\colon M_2(\mathbf{Q}_p) \to B_p$ be an approximation of ι^{-1} to p-adic precision $2m$ relative to the orders $M_2(\mathbf{Z}_p)$ and R_p. Define a $\mathbf{Z}$-lattice in B as follows:*

$$\Lambda(u,v) = f(vM_2(\mathbf{Z}_p)u^*) \cap R + p^{2m+1}R.$$

Then $\operatorname{Hom}_\Gamma(u,v)$ is nonempty if and only if the shortest vectors in $\Lambda(u,v)$ have reduced norm p^{2m}.

Proof. First note that elements in $\Lambda(u,v)$ have reduced norm of valuation at least $2m$. This is so because ι transforms the reduced norm of B_p into the determinant of $M_2(\mathbf{Q}_p)$.

Let $g \in \operatorname{Hom}_\Gamma(u,v)$ and write

$$g = p^{n-m}vxu^* = \iota(y),$$

where $x \in M_2(\mathbf{Z}_p)$ and $y \in p^n\Gamma_n$. We claim that the element $\lambda = p^{m-n}f(g)$ is a shortest vector in $\Lambda(u,v)$ of reduced norm p^{2m}. The first paragraph implies that the reduced norm has to be at least p^{2m}, and since λ has reduced norm p^{2m} it is necessarily shortest. It remains to show that λ belongs to $\Lambda(u,v)$.

Consider $\lambda' = \iota^{-1}(p^{m-n}g)$ and note that λ' belongs to R: clearly $\lambda' = p^{m-n}y$ belongs to $R[1/p]$. But also $\lambda' = \iota^{-1}(vxu^*)$ is p-integral, since vxu^* is so, and ι preserves p-integrality. Therefore

$$\lambda' \in R[1/p] \cap {R^{\max}}_p = R.$$

Thus, by definition of f, we see that λ' and λ are congruent modulo p^{2m+1} and we can write

$$\lambda = \lambda' + p^{2m+1}\alpha, \quad \alpha \in R.$$

This shows that λ' belongs to $\Lambda(u,v)$, and therefore so does λ.

Conversely suppose that $\lambda \in \Lambda(u,v)$ is of reduced norm p^{2m}. Write λ as

$$\lambda = f(vxu^*) + p^{2m+1}\alpha, \quad x \in \mathrm{GL}_2(\mathbf{Z}_p),\ \alpha \in R.$$

Again, by definition of f, we can rewrite the above expression as

$$\lambda = \iota^{-1}(vxu^*) + p^{2m+1}\alpha', \quad \alpha' \in R.$$

We claim that $\iota(\lambda)$ belongs to $\mathrm{Hom}_\Gamma(u,v)$. First, note that $\iota(\lambda/p^m)$ is indeed an element of Γ. It remains to show that $\iota(\lambda)$ takes the vertex corresponding to u to that corresponding to v. For this we write

$$\iota(\lambda) = vxu^* + p^{2m+1}\iota(\alpha') = v(x + p^{2m+1}v^{-1}\iota(\alpha')(\det u)^{-1}u)u^*,$$

and note that the matrix

$$x + p^{2m+1}v^{-1}\iota(\alpha')(\det u)^{-1}u = x + p(p^b v^{-1})\iota(\alpha')u$$

belongs to $\mathrm{GL}_2(\mathbf{Z}_p)$ because x does. Therefore it stabilizes the vertex v_0, and this concludes the proof. □

Remark 3.5. If u and v are ordered edges of $\mathcal{T}$, then the exact analogue of Lemma 3.4 is true with the $\mathbf{Z}$-lattice $\Lambda(u,v)$ replaced with

$$\Lambda'(u,v) = f(v\Gamma_0(p)u^*) \cap R + p^{2m+1}R.$$

Remark 3.6. Lemma 3.4 is useful since the lattice $\Lambda(u,v)$ contains the lattice $p^{2m+1}R$, and so $\Lambda(u,v)$ can be described on a computer using p-adic approximations for a basis, as long as the approximations have at least $2m$ digits of p-adic precision. One may then use standard techniques, like the LLL algorithm as explained in [Coh93, Section 2.6], to find the shortest vectors in $\Lambda(u,v)$. These algorithms have a complexity of $O(m^3)$. This yields an efficient algorithm for determining whether vertices or edges of $\mathcal{T}$ are equivalent under Γ.

3.2. **Computing quotients of the tree.** In this subsection edge always refers to an *ordered edge*. If E is a set of edges of $\mathcal{T}$ and if v is a vertex, we denote by $E(v)$ the subset of those edges of E having origin at v.

We have described how to determine whether edges of $\mathcal{T}$ are Γ-equivalent. Given this, it is not much more difficult to compute the quotient $\Gamma\backslash\mathcal{T}$. The algorithm 1 which we describe below in fact computes a fundamental domain for the action of Γ on $\mathcal{T}$, and this data is richer than the data of the quotient graph. In analogy with the case of Riemann surfaces uniformized by $\mathcal{H}$, a *fundamental domain* in $\mathcal{T}$ for the action of Γ consists of:

(1) a connected subtree $\mathcal{D} \subseteq \mathcal{T}$ whose edges form a set of distinct coset representatives for the action of Γ on the edges of $\mathcal{T}$;
(2) a collection of tuples (u, v, γ) where u and v are distinct boundary vertices of $\mathcal{D}$, and $\gamma \in \Gamma$ satisfies $\gamma u = v$. We compute one such tuple for each pair of identified boundary vertices. The data of all the tuples is refered to as a *boundary pairing*;
(3) in the case when Γ is not Schottky, a list of all the nontrivial edge and vertex stabilizers.

Remark 3.7. Let $\mathcal{D} \subseteq \mathcal{T}$ denote a connected subtree whose edges form a complete set of distinct coset representatives for the action of Γ on the edges of $\mathcal{T}$. If u is a boundary vertex of $\mathcal{D}$, then it must be Γ-equivalent to at least one other boundary vertex. If u were not equivalent to any other boundary vertices, then the edges of $\mathcal{T}$ adjacent to u and outside $\mathcal{D}$ could not be Γ-equivalent to any edges in $\mathcal{D}$, which contradicts the fact that the edges of $\mathcal{D}$ contain a full set of coset representatives.

Algorithm 1 Compute a fundamental domain for Γ acting on $\mathcal{T}$

Input: A prime p, an order $R \subseteq B$ as above, and a splitting $\iota_p \colon R_p \cong M_2(\mathbf{Z}_p)$.
Output: A fundamental domain together with an edge pairing.

```
Select a vertex v_0 ∈ V(T) to begin.
Initialize a queue W with v_0.
Initialize E and P as empty lists.
while W ≠ ∅ do
  Pop v from W.
  for e ∈ E(T)(v) do
    if there is no e' ∈ E(v) which is Γ-equivalent to e then
      Append e to E.
      if there is a vertex v' ∈ W which is Γ-equivalent to t(e) then
        Append (t(e), v', γ) to P, where γ ∈ Γ satisfies γt(e) = v'.
      else
        Push t(e) onto W.
      end if
    end if
  end for
end while
return E, P.
```

Remark 3.8. If one is interested in computing Γ-invariant harmonic cocycles on $\mathcal{T}$, then one can facilitate this by storing extra data during the execution of the fundamental domain algorithm. More precisely, to compute the harmonicity relations, one needs the data of how all of the edges leaving a given vertex get identified under Γ. Algorithm 1 computes this data when testing for membership in the fundamental domain, but with the algorithm stated as is, the data is then discarded. In practice one can store these identifications and associate them with the corresponding vertices of the fundamental domain.

Proposition 3.9. *Algorithm 1 terminates.*

Proof. This follows from the finiteness and connectedness of the quotient graph $\Gamma\backslash\mathcal{T}$. The algorithm incrementally explores a fundamental domain, and produces a boundary pairing for the boundary vertices. □

We wish to analyze the complexity of algorithm 1 in terms of the genus g of the corresponding Shimura curve. Given integer p, N^- and N^+ as above, denote by $X_0(p,N^-,N^+)$ the Shimura curve associated to an Eichler order of level N^+ in the indefinite quaternion algebra with discriminant pN^-. The following genus formula is due to Ogg, and also appears in [GR06, Prop 5.2]:

Theorem 3.10. *The genus of $X_0(p,N^-,N^+)$ is:*

$$g = 1 + \frac{pN^-N^+}{12}\prod_{\ell \mid pN^-}\left(1-\frac{1}{\ell}\right)\prod_{\ell\mid N^+}\left(1+\frac{1}{\ell}\right) - \frac{e_3}{3} - \frac{e_4}{4},$$

where for $k=3,4$:

$$e_k = \prod_{\ell\mid pN^-}\left(1-\left(\frac{-k}{\ell}\right)\right)\prod_{\ell\| N^+}\left(1+\left(\frac{-k}{\ell}\right)\right)\prod_{\ell^2\mid N^+}\nu_\ell(k),$$

$$\nu_\ell(k) = \begin{cases} 2 & \text{if } \left(\frac{-k}{\ell}\right)=1 \\ 0 & \text{else.}\end{cases}$$

Here $\left(\frac{\cdot}{\cdot}\right)$ stands for the Kronecker quadratic symbol.

Proof. See [Ogg83, pg. 280,301]. □

For simplicity, assume that the group Γ is Schottky.

Proposition 3.11. *Let $n = n(p,N^-,N^+)$ be the number sets of the form $\mathrm{Hom}_\Gamma(u,v)$ that need to be computed in order to find a fundamental domain for Γ acting on $\mathcal{T}$. Then*

$$n \le \frac{(g-1)(p^3-2p+2g-1)}{(p-1)^2}.$$

Proof. Let V (respectively E) denote the number of vertices (respectively edges) in a fundamental domain. The algorithm terminates after doing at most pE comparisons among edges, and at most $\frac{V(V-1)}{2}$ comparisons among vertices. Therefore:

$$n = pE + \frac{V(V-1)}{2}.$$

Since we are assuming that Γ is Schottky, the quotient $\Gamma\backslash\mathcal{T}$ is $(p+1)$-regular, and E and V are related by the formula

$$2E = (p+1)V,$$

from which we can extract:

$$V = \frac{2(g-1)}{p-1},\quad E = \frac{(p+1)(g-1)}{p-1}.$$

We thus obtain

$$n \le \frac{(g-1)(p^3-2p+2g-1)}{(p-1)^2}.$$

□

Remark 3.12. The graph $\Gamma'\backslash\mathcal{T}$ (where $\Gamma' \subseteq \Gamma$ is as in the proof above) is Ramanujan (see [Lub10, Theorem 7.3.1]) and this implies, according to [Lub10, Prop 7.3.11], that the diameter of the graph $\Gamma\backslash\mathcal{T}$ is $O(\log g/\log p)$. Therefore the running time of algorithm 1 is

$$O\left(\frac{(\log g)^3(g-1)(p^3-2p+2g-1)}{(\log p)^3(p-1)^2}\right) = O\left(\frac{(\log g)^3 g(p^3+2g)}{(\log p)^3p^2}\right).$$

Remark 3.13. One can reduce to the Schottky case by using the second fact in Proposition 2.4, and the running time in the general case is easily seen to be

$$O\left(\frac{(\log g)^3(Mg-1)(p^3-2p+2Mg-1)}{(\log p)^3(p-1)^2}\right) = O\left(\frac{(\log Mg)^3 Mg(p^3+2Mg)}{(\log p)^3p^2}\right),$$

where M is a positive integer depending only on pN^-, as appearing in Proposition 2.4.

In practice certain improvements can be made that speed up the computation of a fundamental domain in the generic case where few vertices have nontrivial stabilizers (e.g. Schottky groups). If a vertex v of $\mathcal{T}$ has a trivial stabilizer in Γ, then all the edges adjacent to v are necessarily inequivalent under Γ, and so one need not test for membership in a fundamental domain: if one accepts the vertex in the domain, then one must also add all of the adjacent edges to the domain. In the case where Γ is Schottky, one can use this improvement to avoid making *any* edge comparisons in Algorithm 1. The running time then improves to

$$O\left(\frac{(\log g)^3 g^2}{(\log p)^3p^2}\right).$$

Said in terms of the quotient graph, in this speed-up we are using the fact that if Γ is Schottky, then $\Gamma\backslash\mathcal{T}$ is a $(p+1)$-regular connected multigraph without loops. Since for fixed pN^- all but finitely many choices of N^+ produce Schottky groups, this speed-up is significant.

4. Examples

We conclude with some examples of quotient graphs

$$\Gamma(p, N^-, N^+)\backslash\mathcal{T}.$$

As a first example we take $p = 2$, $N^- = 13$ and $N^+ = 1$. The corresponding graph is pictured in Figure 1. This graph corresponds to the special fiber at 2 of the Shimura curve of level 1 constructed from the rational quaternion algebra of discriminant 26. The genus of this curve is 2, as one easily deduces from the graph.

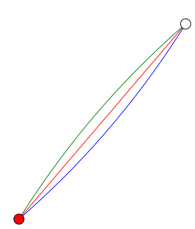

Figure 1. $\Gamma(2, 13, 1)\backslash\mathcal{T}$

In our next example, Figure 2, we increase the level and consider $\Gamma(2, 13, 9)$. The quotient map

$$\Gamma(2, 13, 9)\backslash\mathcal{T} \to \Gamma(2, 13, 1)\backslash\mathcal{T}$$

is described by the colours of the edges of $\Gamma(2, 13, 9)\backslash\mathcal{T}$. Note that this case treats a Schottky group, and so the graph $\Gamma(2, 13, 9)\backslash\mathcal{T}$ is 3-regular. It corresponds to a curve of genus 13.

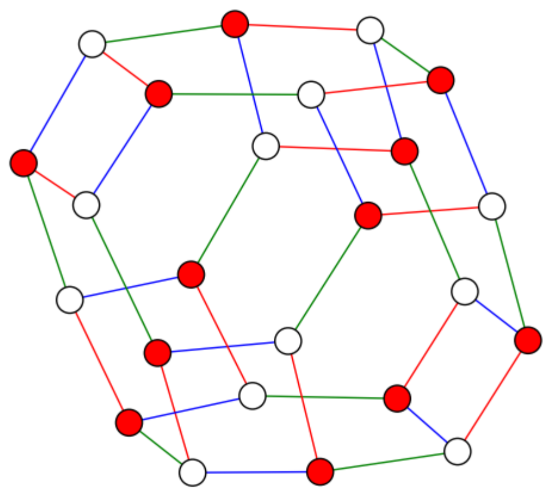

FIGURE 2. $\Gamma(2, 13, 9)\backslash\mathcal{T}$

Next we present a more complicated example, $\Gamma(53, 11, 2)\backslash\mathcal{T}$, which corresponds to the special fiber at 53 of a Shimura curve of genus 31. Note that due to the large number of edges between vertices, rather than graphing all edges we have simply labeled each edge in Figure 3 to indicate the number of edges that pass between adjacent vertices of $\Gamma(53, 11, 2)\backslash\mathcal{T}$. This example took less than 5 seconds to compute on an *Intel Core i5* running at 3.2 GHz.

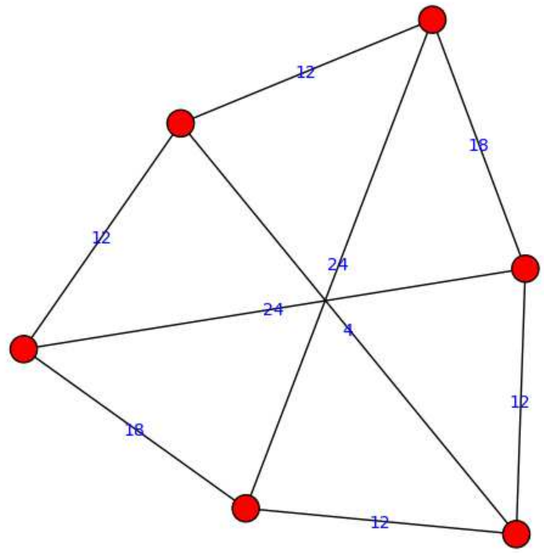

FIGURE 3. $\Gamma(53, 11, 2)\backslash\mathcal{T}$

Finally, by way of comparison, the quotient $\Gamma(211, 1511, 1)\backslash\mathcal{T}$ is a graph with 254 vertices and 26678 edges and corresponds to a curve of genus 26425. It took less than 3 hours to compute on the same *Intel Core i5*.

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